

GAUSSIAN PROCESSES, BRIDGES AND MEMBRANES EXTRACTED FROM SELF-SIMILAR RANDOM FIELDS

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ABSTRACT. We consider the class of selfsimilar Gaussian generalized random fields introduced in Dobrushin [7]. These fields are indexed by Schwartz functions on \mathbb{R}^d and parametrized by a self-similarity index and the degree of stationarity of their increments. We show that such Gaussian fields arise in explicit form by letting Gaussian white noise, or Gaussian random balls white noise, drive a shift and scale shot-noise mechanism on \mathbb{R}^d , covering both isotropic and anisotropic situations. In some cases these fields allow indexing with a wider class of signed measures, and by using families of signed measures parametrized by the points in euclidean space we are able to extract pointwise defined Gaussian processes, such as fractional Brownian motion on \mathbb{R}^d . Developing this method further, we construct Gaussian bridges and Gaussian membranes on a finite domain, which vanish on the boundary of the domain.

1. INTRODUCTION

The main purpose of this work is to propose a method for constructing a variety of Gaussian random processes on \mathbb{R}^d by pointwise evaluation of Gaussian selfsimilar random fields. We will work with zero mean Gaussian fields X defined with respect to Schwartz functions \mathcal{S} on \mathbb{R}^d or, more generally, with respect to a class of signed measures \mathcal{M} on the Borel sets $\mathcal{B}(\mathbb{R}^d)$, writing $\varphi \mapsto X(\varphi)$, $\varphi \in \mathcal{S}$, and $\mu \mapsto X(\mu)$, $\mu \in \mathcal{M}$. Defining the dilations φ_c of φ and μ_c of μ , by

$$\varphi_c(x) = c^{-d} \varphi(c^{-1}x), \quad x \in \mathbb{R}^d, \quad \mu_c(A) = \mu(c^{-1}A), \quad A \subset \mathcal{B}(\mathbb{R}^d),$$

a random field is said to be selfsimilar with self-similarity index H , if

$$X(\varphi_c) \stackrel{d}{=} c^H X(\varphi), \quad X(\mu_c) \stackrel{d}{=} c^H X(\mu), \quad c > 0.$$

Dobrushin [7], pioneered a theory of generalized random fields with r th order stationary increments, and characterized all Gaussian selfsimilar random fields on \mathbb{R}^d by providing a representation of the covariance functional $C(\varphi, \psi) = \text{Cov}(X(\varphi), X(\psi))$ parametrized by r and H . We will use special instances of such random fields $\mu \mapsto X(\mu)$ with $H > 0$ and extract Gaussian processes $(X_t)_{t \in \mathbb{R}^d}$ by putting $X_t = X(\mu_t)$ for a suitably chosen family of indexing measures $(\mu_t)_{t \in \mathbb{R}^d}$.

Gaussian white noise $M_d(dx)$, which is the case $r = 0$ and $H = -d/2$, is such that $M_d(\varphi)$ is a zero mean Gaussian random field with covariance $C(\varphi, \psi) = \int_{\mathbb{R}^d} \varphi(x) \psi(x) dx$. Gaussian random balls white noise is a class of isotropic, generalized random fields W_β , such that for a suitable family \mathcal{M}_β of signed measures,

$$W_\beta(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu(B(x, u)) M_\beta(dx, du), \quad \mu \in \mathcal{M}_\beta,$$

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where $B(x, u)$ is the Euclidean ball centered in x with radius u and $M_\beta(dx, du)$ is Gaussian white noise on $\mathbb{R}^d \times [0, \infty)$ with control measure $\nu(dx, du) = dx u^{-\beta-1} du$. Such fields are known to be well-defined for $d-1 < \beta < d$ and $d < \beta < 2d$ and W_β is selfsimilar with index $H = (d - \beta)/2 \in (-d/2, 0) \cap (0, 1/2)$, [3],[13]. These classes of selfsimilar random fields may be recognized as the cases $r = 0$, $-d/2 < H < 0$ and $r = 1$, $0 < H < 1/2$, respectively, of isotropic fields in Dobrushin's characterization. By considering the Riesz transform

$$(-\Delta)^{-m/2} \varphi(x) = \int_{\mathbb{R}^d} |x - y|^{-(d-m)} \varphi(y) dy, \quad 0 < m < d,$$

and random fields defined by

$$X(\varphi) = W_\beta((-\Delta)^{-m/2} \varphi),$$

for a suitably restricted class of test functions φ , it is also possible to extend the range of the self-similarity index H covered by random balls models to any value $H \neq \mathbb{Z}$ if $d \geq 2$ and $H \neq \frac{1}{2}\mathbb{Z}$ if $d = 1$, see [3].

In this work we present a more general construction of Gaussian selfsimilar shot noise random fields, which naturally includes anisotropic models. We apply the same Gaussian white noises, M_d and M_β as above, use the method of indexing random fields with a class of signed measures, and extend the range of self-similarity index with the help of the Riesz transform. These tools allow us to build, in particular, Gaussian selfsimilar random fields $\mu \mapsto X(\mu)$ with index $H > 0$, and apply to them a family of measures $(\mu_t)_{t \in \mathbb{R}^d}$. By extracting the random fields in this manner, we obtain pointwise defined random processes

$$t \mapsto Y_t = X(\mu_t), \quad t \in \mathbb{R}^d,$$

which inherit relevant properties from the underlying random fields. The guiding example is fractional Brownian motion $B_H(t)$, $t \in \mathbb{R}^d$, with $0 < H < 1$, which we extract from an appropriate random field by applying $\mu_t = \delta_t - \delta_0$ and/or $\mu_t = (-\Delta)^{-m/2}(\delta_t - \delta_0)$ with a suitable m . As a byproduct we obtain a new representation of fractional Brownian motion in terms of M_β , which may be compared to the well-balanced representation that results from using M_d . To illustrate isotropy and anisotropy in natural situations, we also compare the random balls construction with a random cylinder model, which leads to a comparison between fractional Brownian motions and fractional Brownian sheets.

To investigate further the range of applicability of the briefly explained extraction principle, we consider for the one-dimensional case $d = 1$ construction of Gaussian bridges on an interval of the real line and construction of Volterra processes. In higher dimensions we propose the construction of membranes on a bounded domain D in \mathbb{R}^d , as Gaussian processes X_t , $t \in D$, such that X_t converges in probability to 0 as t tends to ∂D . Finally, we discuss membranes obtained from Gaussian random balls white noise, which is thinned by a hard boundary in the sense that balls that do not fall entirely within the domain are discarded.

Our presentation is organized as follows. In the next Section 2 we give preliminaries on Gaussian random measures and fields including an account of Dobrushin's characterization of selfsimilar random fields. In Section 3 we present our main results on Gaussian shot noise random fields as Theorem 2, devoted to fields generated by a wide range of pulse functions and random balls white noise M_β , and Theorem 3, which instead applies a singular shot function h_β and regular white noise M_d . The discussion on random cylinder models is included as a separate subsection. Section 4 contains our account of the extraction method and the various results on fractional Brownian motion, Gaussian bridges, Volterra processes and membranes constructed by soft boundary thinning of the harmonic measure. Finally, Section 5 is devoted to membranes generated by hard boundary thinning.

2. PRELIMINARIES ON GAUSSIAN RANDOM MEASURES AND FIELDS

Let (D, \mathcal{D}, ν) be a measure space and let $\mathcal{D}_\nu = \{A \in \mathcal{D} : \nu(A) < \infty\}$ denote the set of measurable sets with finite measures. A Gaussian stochastic measure on (D, \mathcal{D}, ν) is a family of centered Gaussian random variables $Z(A)$, $A \in \mathcal{D}_\nu$, such that

$$\text{Cov}(Z(A), Z(B)) = \nu(A \cap B), \quad A, B \in \mathcal{D}_\nu,$$

and the corresponding Gaussian stochastic integral $f \mapsto \int f dZ$ is the linear isometry $f \mapsto \mathcal{I}(f)$ of $L^2(D, \mathcal{D}, \nu)$ into a Gaussian Hilbert space H , defined by $\mathcal{I}(\mathbb{I}_A) = Z(A)$, $A \in \mathcal{D}_\nu$, [12] Ch. 7.2. Our main examples will be the Euclidean case $D = \mathbb{R}^d$ with control measure $\nu(dx)$ which is uniform or absolutely continuous with respect to Lebesgue measure on \mathbb{R}^d , and simple product spaces, such as $D = \mathbb{R}^d \times \mathbb{R}_+$ equipped with a product measure $\nu(dx, du) = dx \nu_\gamma(du)$, where $\nu_\gamma(du) = u^{-\gamma-1} du$ is a power law measure on the real positive line.

Gaussian white noise on \mathbb{R}^d . We denote by $M_d(dx)$ the Gaussian stochastic measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$, the d -dimensional Euclidean space with the Borel σ -algebra $\mathcal{B}(\mathbb{R}^d)$ and Lebesgue control measure dx . The stochastic integral with respect to M_d is the linear map $f \mapsto \mathcal{I}(f) = \int_{\mathbb{R}^d} f(x) M_d(dx)$ defined as an isometry from $L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$, equipped with the inner product norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$, where $\langle f, g \rangle = \int f g dx$, into a Gaussian space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Let \mathbb{E} be the expectation operator associated with \mathbb{P} . Since

$$\int_{\mathbb{R}^d} f(x) M_d(dx) \int_{\mathbb{R}^d} g(y) M_d(dy) = \int_{\mathbb{R}^d} f(x) g(x) dx,$$

the covariance functional $\mathbb{E}(\mathcal{I}(f) \mathcal{I}(g)) = \langle f, g \rangle$, is given by the ordinary inner product of L^2 functions. The same construction works in greater generality, such as anisotropic white noise with control measure $w(x) dx$ for a nonnegative weight function w and covariance functional given by the inner product of the weighted space $L_2(\mathbb{R}^d, w dx)$.

Gaussian Hilbert space. Let \mathcal{S} be the space of real, rapidly decreasing and smooth Schwartz functions on \mathbb{R}^d . The continuous, bilinear form $\langle \cdot, \cdot \rangle$ is symmetric, semi-definite and non-degenerate on \mathcal{S} . Hence $(\mathcal{S}, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space with inner product $\langle f, g \rangle$ for which the completion to a Hilbert space is the usual space $L^2(\mathbb{R}^d)$ of real-valued square-integrable functions on \mathbb{R}^d . Also, by Minlos's theorem, $\langle f, g \rangle$ corresponds to a unique Gaussian measure \mathbb{P} on the space \mathcal{S}' of real tempered distributions, the dual space of \mathcal{S} . Indeed, we obtain a Gaussian Hilbert space $H \subset L^2(\mathbb{P})$ such that the linear functional $f \mapsto u(f)$ on \mathcal{S}' is an isometry which defines the Gaussian white noise measure on \mathcal{S}' . As a Gaussian field on an L^2 -space, white noise on generalized Schwartz distributions may be regarded as the stochastic integral $f \mapsto \int f(x) M_d(dx)$, cf. [12], Ex. 1.16, Ex. 7.24.

Stationary Gaussian random fields. We write $|j| = \sum_{k=1}^d j_k$ for each d -dimensional multi-index $j = (j_1, \dots, j_d)$ and $x^j = \prod_{k=1}^d x_k^{j_k}$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, and consider the sequence \mathcal{S}_r , $r = 0, 1, \dots$, of closed subspaces of \mathcal{S} , such that

$$\mathcal{S}_r = \left\{ \varphi \in \mathcal{S} : \int_{\mathbb{R}^d} x^j \varphi(x) dx = 0, |j| < r \right\}, \quad r = 1, 2, \dots, \quad \mathcal{S}_0 = \mathcal{S}.$$

A Gaussian random field over \mathcal{S}_r is a continuous, linear functional $X : \mathcal{S}_r \rightarrow \mathbb{R}$, such that $X(\varphi)$ is a Gaussian random variable for each $\varphi \in \mathcal{S}_r$. The field is said to be isotropic if the distribution is invariant under rotations of \mathbb{R}^d and stationary if it is invariant under translations. A Gaussian random field X over \mathcal{S}_0 is said to have stationary r th increments if the restriction of X to \mathcal{S}_r is a stationary Gaussian random field over \mathcal{S}_r . Let \mathcal{E} be the

symmetric semidefinite bilinear form on \mathcal{S}_r , defined by $\mathcal{E}(\varphi, \psi) = \mathbb{E}X(\varphi)X(\psi)$. Then $(\mathcal{S}_r, \mathcal{E})$ is a pre-Hilbert space with inner product $\mathcal{E}(\varphi, \psi)$, which may be completed to a Hilbert space $\mathcal{S}_\mathcal{E}$ with norm $\sqrt{\mathcal{E}(\varphi, \varphi)}$, and then $\varphi \mapsto X(\varphi)$ is an isometry of $\mathcal{S}_\mathcal{E}$ onto a Gaussian Hilbert space in \mathcal{S}'_r . Conversely, by Minlos's theorem, any continuous bilinear semidefinite symmetric form gives rise to a unique Gaussian field on \mathcal{S}'_r .

More generally, we may consider Gaussian random fields defined on a space of measures. Let $(\mathcal{M}, \|\cdot\|)$ denote the normed space of signed measures μ on \mathbb{R}^d with variation measure $|\mu|$, such that the total variation norm is finite, $\|\mu\| = |\mu|(\mathbb{R}^d) < \infty$. We put $\mathcal{M}_0 = \mathcal{M}$ and for $r = 1, 2, \dots$,

$$(1) \quad \mathcal{M}_r = \left\{ \mu \in \mathcal{M} : \int_{\mathbb{R}^d} |x|^{r-1} |\mu|(dx) < \infty, \int_{\mathbb{R}^d} x^j \mu(dx) = 0, |j| < r \right\}.$$

The subspaces \mathcal{M}_r are closed under translations $\mu(A) \mapsto \mu(A - s)$, $s \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$. In this framework a Gaussian random field X over \mathcal{M}_r is defined in analogy to those over \mathcal{S}_r , and the notions of isotropy, translation invariance and r th order stationary increments carry over. Moreover, by completion one obtains a Gaussian Hilbert space $\mathcal{M}_\mathcal{E}$ and an isometry $\mu \mapsto X(\mu)$ onto a Gaussian Hilbert space in the dual space of distributions, cf. [12] Def. 1.18, and [3] Sect. 3.1.

The M. Riesz potential kernel. Let $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_d^2$ be the usual Laplacian operator on \mathbb{R}^d . The Fourier transform $\widehat{\Delta\varphi}$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, satisfies

$$\widehat{\Delta\varphi}(\xi) = -|\xi|^2 \widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $m \in \mathbb{Z}$, the power operators $(-\Delta)^{-m/2}$ of the Laplace operator may be defined formally using the Fourier transform, by

$$(2) \quad \widehat{(-\Delta)^{-m/2}\varphi}(\xi) = |\xi|^{-m} \widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^d.$$

In the context of random fields the family of operators $(-\Delta)^{-m/2}$, $m \in \mathbb{Z}$, can be given a precise meaning as linear homeomorphisms defined on the intersection space $\mathcal{S}_\infty = \cap_{r \geq 0} \mathcal{S}_r$, see [3]. For $1 \leq m \leq d-1$, and more generally for a non-integer parameter m , $0 < m < d$, the application $(-\Delta)^{-m/2}\varphi$ is well-defined for $\varphi \in \mathcal{S}$ and can be realized as a fractional integral with respect to the Riesz kernel, given by

$$(-\Delta)^{-m/2}\varphi(x) = C_{m,d} \int_{\mathbb{R}^d} |x-y|^{-(d-m)} \varphi(y) dy, \quad C_{m,d} = \frac{\Gamma((d-m)/2)}{\pi^{d/2} 2^m \Gamma(m/2)}.$$

In one dimension, $d = 1$, this extends naturally by putting $(-\Delta)^{-1/2}\varphi(x) = \int_x^\infty \varphi(y) dy$.

For signed measures in $\mu \in \mathcal{M}$ we will understand $(-\Delta)^{-m/2}\mu$ to be the map generated by the Riesz potential of order m , defined by

$$(-\Delta)^{-m/2}\mu(dx) = C_{m,d} \int_{\mathbb{R}^d} |x-y|^{-(d-m)} \mu(dy) dx.$$

For the one-dimensional case, $(-\Delta)^{-1/2}\mu(dx) = \int_x^\infty \mu(dy) dx$. The Riesz potential of order m is finite almost everywhere if and only if [15]

$$\int_{\{y \in \mathbb{R}^d : |y| > 1\}} \frac{\mu(dy)}{|y|^{d-m}} < \infty,$$

and this condition will be satisfied for all measures μ considered here. With regards to the Riesz kernel we will make frequent use of the composition rule

$$(3) \quad \int_{\mathbb{R}^d} \frac{C_{m_1,d}}{|y-x|^{d-m_1}} \frac{C_{m_2,d}}{|y'-x|^{d-m_2}} dx = \frac{C_{m_1+m_2,d}}{|y-y'|^{d-m_1-m_2}},$$

valid for $0 < m_1, m_2 < d$, $m_1 + m_2 < d$.

Selfsimilar Gaussian random fields. For $\varphi \in \mathcal{S}$ let φ_c be the dilation defined by $\varphi_c(x) = c^{-d}\varphi(c^{-1}x)$, $c \geq 0$. Clearly, $\varphi_c \in \mathcal{S}_r$ if $\varphi \in \mathcal{S}_r$. A random field X over \mathcal{S}_r is said to be selfsimilar with index H , or H -selfsimilar, if $X(\varphi_c)$ has the same distribution as $c^H X(\varphi)$, $\varphi \in \mathcal{S}_r$. Similarly, for $\mu \in \mathcal{M}(\mathbb{R}^d)$ define μ_c by $\mu_c(B) = \mu(B/c)$, $B \in \mathcal{B}(\mathbb{R}^d)$. We will sometimes write $\mu \mapsto X(\mu)$ for the mapping of a random field even if the space of measures coincides with the absolutely continuous signed measures $\mu(dx) = \varphi(x) dx$, $\varphi \in \mathcal{S}_r$. In this notation, a random field is H -selfsimilar if $X(\mu_c)$ has the same distribution as $c^H X(\mu)$, for all relevant μ .

Theorem 1 (Dobrushin '79 [7]). *Fix $r \geq 0$. A Gaussian random field X on \mathcal{S}_r is stationary and H -selfsimilar if and only if the covariance functional $C(\varphi, \psi) = \text{Cov}(X(\varphi), X(\psi))$ is given by*

$$\begin{aligned} C(\varphi, \psi) = & \sum_{|j|=|k|=r} a_{jk} \int_{\mathbb{R}^d} x^j \varphi(x) dx \int_{\mathbb{R}^d} y^k \psi(y) dy \\ & + \int_{\mathbb{S}^{d-1}} \int_0^\infty \widehat{\varphi}(u\theta) \overline{\widehat{\psi}(u\theta)} u^{-2H-1} du \sigma(d\theta), \end{aligned}$$

where the matrix (a_{jk}) is symmetric and nonnegative definite and $\sigma(d\theta)$ is a finite, positive, and reflection-invariant measure on the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d . Here, if $H > r$ then $X \equiv 0$, if $H = r$ then $\sigma(d\theta) = 0$ and if $H < r$ then $(a_{jk}) = 0$.

Random polynomials. The special case $H = r$ in Theorem 1 corresponds to random polynomials. For $x \in \mathbb{R}^d$ let $X_r(x)$ be the Gaussian random polynomial of order r defined by

$$X_r(x) = \sum_{|j| \leq r} \xi_j x^j,$$

where $x^j = \prod_{k=1}^d x_k^{j_k}$ for each multi index $j = (j_1, \dots, j_d)$, $|j| = \sum_{k=1}^d j_k$, and (ξ_j) are standard Gaussian random variables. Then

$$X_r(\varphi) = \sum_{|j| \leq r} \xi_j \int_{\mathbb{R}^d} x^j \varphi(x) dx, \quad \varphi \in \mathcal{S},$$

defines a corresponding Gaussian random field on \mathcal{S} . By restricting to \mathcal{S}_r one obtains the order r terms

$$X_r(\varphi) = \sum_{|j|=r} \xi_j \int_{\mathbb{R}^d} x^j \varphi(x) dx, \quad \varphi \in \mathcal{S}_r.$$

As a field on \mathcal{S}_r the polynomial field X_r is r -selfsimilar and stationary. Indeed, if $\varphi \in \mathcal{S}_r$ then

$$X_r(\varphi(\cdot + a)) = \sum_{|j|=r} \xi_j \int_{\mathbb{R}^d} (x-a)^j \varphi(x) dx = \sum_{|j|=r} \xi_j \int_{\mathbb{R}^d} x^j \varphi(x) dx.$$

Nondegenerate selfsimilar Gaussian random fields. Considering Gaussian H -selfsimilar random fields on \mathcal{S}_r with $H < r$, it follows by Theorem 1 that

$$(4) \quad C(\varphi, \psi) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \widehat{\varphi}(u\theta) \overline{\widehat{\psi}(u\theta)} u^{-2H-1} du \sigma(d\theta), \quad \varphi \in \mathcal{S}_r,$$

and if we specialize to isotropic random fields then the covariance functional takes the form

$$(5) \quad C(\varphi, \psi) = \text{const} \int_{\mathbb{R}^d} \widehat{\varphi}(z) \overline{\widehat{\psi}(z)} |z|^{-2H-d} dz.$$

The most basic case is $H = -d/2$ and $r = 0$ combined with a rotationally symmetric measure $\sigma(d\theta)$. By Parseval's identity, this is Gaussian white noise $M_d(dx)$ with $M_d(\varphi) \sim N(0, \int \varphi(x)^2 dx)$ and $C(\varphi, \psi) = \int_{\mathbb{R}^d} \varphi(x) \psi(x) dx$ (ignoring constants).

If we return to (4) but restrict the range of parameters to $-d/2 < H < r$, then the covariance may be recast into

$$(6) \quad C(\varphi, \psi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x) \psi(y) |x - y|^{2H} \mathcal{K}\left(\frac{x - y}{|x - y|}\right) dx dy,$$

where \mathcal{K} is an anisotropy weight function on \mathbb{S}^{d-1} defined by

$$\mathcal{K}(e) = \int_{\mathbb{S}^{d-1}} \int_0^\infty e^{-ir\theta \cdot e} u^{-2H-1} du \sigma(d\theta), \quad e \in \mathbb{S}^{d-1}.$$

Recalling from (1) the setting of indexing measures in \mathcal{M}_r we conclude that, with the exception of independently scattered white noise, all isotropic selfsimilar Gaussian random fields are characterized by a covariance functional $C(\mu, \mu') = \text{Cov}(X(\mu), X(\mu'))$, such that

$$(7) \quad C(\mu, \mu') = \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^{2H} \mu(dx) \mu'(dy), \quad \mu, \mu' \in \widetilde{\mathcal{M}}.$$

For $-d/2 < H < 0$ the relevant set $\widetilde{\mathcal{M}} \subset \mathcal{M}_r$, consists of signed measures with finite Riesz-energy. For $0 < H < r$ the moment condition $\int \mu(dy) = 0$ enters and we have the additional representation

$$C(\mu, \mu') = \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} (|x|^{2H} + |y|^{2H} - |x - y|^{2H}) \mu(dx) \mu'(dy).$$

The self-similarity of the model is equivalent to the second order self-similarity property $C(\mu_c, \mu_c) = c^{2H} C(\mu, \mu)$. Our final remark in this section is that because of (2) and (4), the Riesz kernel preserves self-similarity, in the sense

$$(8) \quad C((-\Delta)^{-m/2} \varphi, (-\Delta)^{-m/2} \psi) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \widehat{\varphi}(r\theta) \overline{\widehat{\psi}(r\theta)} u^{-2H-2m-1} du \sigma(d\theta).$$

Thus, if $X(\varphi)$ is selfsimilar with index H then the random field $Y(\varphi)$ defined by $Y(\varphi) = X((-\Delta)^{-m/2} \varphi)$ for some m with $H + m < r$, is selfsimilar with index $H+m$, cf. [3] Thm 4.7.

3. GAUSSIAN SHOT NOISE RANDOM FIELDS

In this section we introduce a wide class of Gaussian selfsimilar random fields on \mathbb{R}^d , generated by white noise and obtained by a shot noise construction. Isotropic as well as anisotropic models are covered. The white noise is defined on the extended space $\mathbb{R}^d \times \mathbb{R}_+$ where the additional degree of freedom may be thought of as a random radius of an euclidean ball located in \mathbb{R}^d . A class of nonnegative functions in $L_2(\mathbb{R}^d)$ adds further generality to the model, acting as pulse functions for a shot noise mechanism driven by the random balls. The Riesz kernel transform furthermore provides means of moving from one range of self-similarity

indices to another. In the end all combined we obtain efficient methods to extract a variety of processes, bridges and membranes from these Gaussian random fields.

Random ball white noise. For fixed spatial dimension $d \geq 1$ we consider a parameter β , such that

$$\beta \in (d-1, d) \cup (d, 2d),$$

put

$$(9) \quad \tilde{\nu}_\beta(du) = u^{-\beta-1} du, \quad u > 0, \quad \nu(dz) = dx \tilde{\nu}_\beta(du), \quad z = (x, u) \in \mathbb{R}^d \times \mathbb{R}_+,$$

and let $M_\beta(dz)$ be white noise on $\mathbb{R}^d \times \mathbb{R}_+$ defined by the control measure $\nu(dz)$. Also, with some abuse of notation, we write $M_d(dz)$ for Gaussian noise with control measure $\nu(dz) = dx \delta_1(du)$, which in this manner is identified with Gaussian white noise $M_d(dx)$ as introduced earlier. It is convenient to let each Gaussian point (x, u) represent a Euclidean ball $B(x, u)$ in \mathbb{R}^d centered in x with radius $u > 0$. The general method of evaluating random fields that we adopt in this work amounts to measure the aggregation of Gaussian mass from all of $M_\beta(dz)$ as the stochastic integral

$$(10) \quad X(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu(B(x, u)) M_\beta(dz),$$

where μ belongs to a suitable class of signed measures. This approach is introduced in [13] and developed further in [3] and [4].

As a preparation to help see the origin of self-similarity in these models we begin with the simplest case of fixed size balls corresponding to $M_d(dz)$, and consider

$$X(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu(B(x, u)) M_d(dz) = \int_{\mathbb{R}^d} \mu(B(x, 1)) M_d(dx).$$

This model is Gaussian with covariance functional

$$C(\mu, \mu') = \int_{\mathbb{R}^d} \mu(B(x, 1)) \mu'(B(x, 1)) dx = \int_{\mathbb{R}^d \times \mathbb{R}^d} |B(y, 1) \cap B(y', 1)| \mu(dy) \mu'(dy').$$

The volume $V(|y - y'|) = |B(y, 1) \cap B(y', 1)|$ of two intersecting balls only depends on the distance between the center points y and y' and is given by

$$(11) \quad V(u) = 2v_{d-1} \int_{u/2}^1 (1 - s^2)^{\frac{d-1}{2}} ds, \quad u \leq 2,$$

and $V(u) = 0$ for $u > 2$, where $v_d = |B(0, 1)|$ is the volume of the unit ball in \mathbb{R}^d , see [11]. The one-point evaluations

$$X(\delta_t) = \int_{\mathbb{R}^d} \mathbb{I}_{\{|x-t| \leq 1\}} M_d(dx), \quad t \in \mathbb{R}^d,$$

exist and generate a point-wise defined zero mean Gaussian random field with covariance $C(\delta_t, \delta_{t'}) = V(|t - t'|)$. This random field does not possess the self-similarity property itself but if we replace the control measure $dx \delta_1(du)$ with $dx \tilde{\nu}_\beta(du)$ for $M_\beta(dz)$ in (10), then the

covariance is

$$\begin{aligned} C(\mu, \mu') &= \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu(B(x, u)) \mu'(B(x, u)) dx u^{-\beta-1} du \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_0^\infty u^d V(|y - y'|/u) u^{-\beta-1} du \mu(dy) \mu'(dy') \\ &= \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\mu(dy) \mu'(dy')}{|y - y'|^{\beta-d}}, \end{aligned}$$

which is selfsimilar with index $H = (d - \beta)/2$ according to (5), assuming μ and β are such that the integral exists. As a second type of modification we replace $\mu(B(x, 1))$ in the previous expression for $X(\mu)$ with integration of μ with respect to a spatially shifted power law function $h_\gamma(y) = |y|^{-(d-\gamma)}$, $0 < \gamma < d/2$, and consider

$$X(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_\gamma(y - x) \mu(dy) M_d(dx).$$

The heuristic picture of randomly sized overlapping balls in \mathbb{R}^d now changes to one of overlapping pulse functions. By (3), the covariance is found to have the selfsimilar shape

$$\text{Cov}(X(\mu), X(\mu')) = \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\mu(dy) \mu'(dy')}{|y - y'|^{d-2\gamma}}.$$

An equivalent interpretation of this particular construction is that we integrate the Riesz kernel with respect to white noise $M_d(dx)$:

$$(12) \quad \langle M_d, (-\Delta)^{-\gamma/2} \mu(\cdot) \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^{-(d-\gamma)} \mu(dy) M_d(dx).$$

We emphasize the distinction between the use of the Riesz kernel in (12) as opposed to the effect of Riesz integration by shifting from μ to $(-\Delta)^{-m/2} \mu$ in the random balls model in (10), applying the composition rule (3), and obtaining

$$\begin{aligned} C((-\Delta)^{-m/2} \mu, (-\Delta)^{-m/2} \mu') &= \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(-\Delta)^{-m/2} \mu(dy) (-\Delta)^{-m/2} \mu'(dy')}{|y - y'|^{\beta-d}} \\ &= \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\mu(dy) \mu'(dy')}{|y - y'|^{\beta-d-2m}}. \end{aligned}$$

The range of the self-similarity index in these relations will depend on a more detailed analysis of which combinations of parameters and admissible measures one can use, and will be part of the subsequent results.

Shot noise. We are now in position to introduce a Gaussian shot noise random field X_h driven by $M_\beta(dz)$ and with a given pulse function h in $L_2(\mathbb{R}^d)$. We define the shift and scale mapping

$$(13) \quad \tau_z h(y) = h((y - x)/u), \quad z = (x, u) \in \mathbb{R}^d \times \mathbb{R}_+,$$

and put

$$X_h(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \langle \mu, \tau_z h \rangle M_\beta(dz), \quad \langle \mu, \tau_z h \rangle = \int_{\mathbb{R}^d} \tau_z h(y) \mu(dy).$$

Occasionally we use τ_x as a short hand notation for $\tau_{(x,1)}$. The construction of the shot noise then relies on stating proper assumptions on the class of measures μ involved and on the class of admissible pulse functions h for which X_h will exist as a Gaussian stochastic integral. The shot noise mechanism we investigate here is inspired by similar constructions in [5].

Following [3], for $\beta \neq d$ we let

$$\mathcal{M}^\beta = \left\{ \mu \in \mathcal{M} : \exists \alpha \text{ s.t. } \alpha < \beta < d \text{ or } d < \beta < \alpha \right. \\ \left. \text{and } \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{d-\alpha} |\mu|(dy) |\mu|(dy') < \infty \right\}.$$

For $d < \beta$ this space of measures is closely related to the set of measures with finite Riesz energy. Then we combine \mathcal{M}^β with the previously introduced sets \mathcal{M}_r , $r = 0, 1, \dots$, and put

$$\mathcal{M}_r^\beta = \mathcal{M}^\beta \cap \mathcal{M}_r. \quad \widetilde{\mathcal{M}}_\beta = \begin{cases} \mathcal{M}^\beta, & d < \beta < 2d, \\ \mathcal{M}_1^\beta, & d-1 < \beta < d. \end{cases}$$

Let \mathcal{H}_β be the subset of functions in $L_2(\mathbb{R}^d)$, such that, for the case $d < \beta < 2d$,

$$\left| \int_{\mathbb{R}^d} h(x) h(x+y) dx \right| \leq \frac{\text{const}}{|y|^{\alpha-d}}, \quad \text{all } y \in \mathbb{R}^d \text{ and } \alpha \in (\beta, 2d),$$

and, for the case $d-1 < \beta < d$,

$$\left| \int_{\mathbb{R}^d} h(x) (h(x+y) - h(x)) dx \right| \leq \text{const} |y|^{d-\alpha}, \quad \text{all } y \in \mathbb{R}^d \text{ and } \alpha \in (d-1, \beta).$$

Theorem 2. Fix $\beta \in (d-1, d) \cap (d, 2d)$. Let $M_\beta(dz)$ be the Gaussian random ball white noise on $\mathbb{R}^d \times \mathbb{R}_+$ with control measure $\nu(dz) = dx \tilde{\nu}_\beta(du)$ as defined in (9). Assume $h \in \mathcal{H}_\beta$ and let H denote the parameter

$$H = \frac{d-\beta}{2} \in \begin{cases} (-d/2, 0), & d < \beta < 2d, \\ (0, 1/2), & d-1 < \beta < d. \end{cases}$$

i) The shot noise random field

$$\mu \mapsto X_h(\mu), \quad \mu \in \widetilde{\mathcal{M}}_\beta,$$

is well-defined as a zero mean Gaussian H -selfsimilar stochastic integral with covariance functional

$$\text{Cov}(X_h(\mu), X_h(\mu')) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{K}_h\left(\frac{y-y'}{|y-y'|}\right) \frac{\mu(dy) \mu'(dy')}{|y-y'|^{\beta-d}},$$

where the kernel function \mathcal{K}_h is defined on the unit sphere \mathbb{S}^{d-1} and given by

$$\mathcal{K}_h(e) = \begin{cases} C_h \int_0^\infty u^{d-1-\beta} \int_{\mathbb{R}^d} h(x) h(x+e/u) dx du, & d < \beta < 2d, \\ C_h \int_0^\infty u^{d-1-\beta} \int_{\mathbb{R}^d} h(x) (h(x+e/u) - h(x)) dx du, & d-1 < \beta < d, \end{cases}$$

$e \in \mathbb{S}^{d-1}$, for some constant C_h . In particular, if h is rotationally symmetric on \mathbb{R}^d then $\mathcal{K}_h(e) = \mathcal{K}_h$ is a constant and the random field X_h is isotropic.

ii) Consider the restricted range $d < \beta < 2d$. For the case $d \geq 2$, let m be a real number such that

$$1 < 2m < d, \quad 0 < d - \beta + 2m < 2,$$

and put $H' = H + m$. Assume $(-\Delta)^{-m/2} \mu \in \mathcal{M}^\beta$. Then the random field

$$\mu \mapsto X_h((-\Delta)^{-m/2} \mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \langle (-\Delta)^{-m/2} \mu, \tau_z h \rangle M_\beta(dz),$$

is H' -selfsimilar. For the one-dimensional case, $d = 1$, with $1 < \beta < 2$, the random field

$$\mu \mapsto X_h((-\Delta)^{-1/2}\mu) = \int_{\mathbb{R} \times \mathbb{R}_+} \int_{\mathbb{R}} h((y-x)/u) \mu([y, \infty)) dy M_\beta(dx, du),$$

is $(3 - \beta)/2$ -selfsimilar for μ such that $\int_y^\infty \mu(dz) dy \in \mathcal{M}^\beta$.

Proof. i) The Gaussian stochastic integral $X_h(\mu)$ exists if and only if the variance

$$\text{Cov}(X_h(\mu), X_h(\mu)) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \langle \mu, \tau_{x,u} h \rangle^2 dx u^{-\beta-1} du$$

is finite. We need to verify that this is the case under the stated assumptions and establish the explicit form of the covariance functional. The proof can be seen as an adaptation of Lemma 2.3 in [3] to the case of a shot noise weight function h .

We begin with the case $d < \beta < 2d$. Then $\widetilde{\mathcal{M}}_\beta = \mathcal{M}^\beta$. We introduce the function g defined by

$$g(u) = \int_{\mathbb{R}^d} \langle \mu, \tau_{x,u} h \rangle^2 dx, \quad u > 0.$$

Using Fubini's theorem and homogeneity, we obtain

$$(14) \quad g(u) = u^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d} h(x) h(x + (y - y')/u) dx \mu(dy) \mu(dy').$$

Since $h \in \mathcal{H}_\beta$ and $\mu \in \mathcal{M}^\beta$ we can find $\alpha \in (\beta, 2d)$, such that

$$(15) \quad 0 < g(u) \leq \text{const } u^\alpha \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\mu|(dy) |\mu|(dy')}{|y - y'|^{\alpha-d}} < \infty.$$

On the other hand, using Hölder's inequality and $\mu \in \mathcal{M}$, it follows from (14) that $g(u) \leq \|h\|_2 \|\mu\|^2 u^d$, so that

$$(16) \quad 0 < g(u) \leq \text{const } \min(u^\alpha, u^d)$$

and hence

$$\int_0^\infty g(u) u^{-\beta-1} du = \int_{\mathbb{R}^d \times \mathbb{R}_+} \langle \mu, \tau_z h \rangle^2 \nu(dz) < \infty.$$

Next we may replace g in the left-hand side integral by the integral expression in (14) and apply a change of variables, to obtain

$$\int_{\mathbb{R}^d \times \mathbb{R}_+} \langle \mu, \tau_z h \rangle^2 \nu(dz) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{K}_h\left(\frac{y - y'}{|y - y'|}\right) \frac{\mu(dy) \mu(dy')}{|y - y'|^{\beta-d}}$$

with the desired function \mathcal{K}_h , as stated in the theorem. By (6), this is the covariance functional for a selfsimilar Gaussian model with self-similarity index $H = -(\beta - d)/2 < 0$.

For the remaining case $d - 1 < \beta < d$ in statement i), we have $\mu \in \mathcal{M}_1$ and hence $\int_{\mathbb{R}^d} \mu(dx) = 0$. Thus, we may replace (14) by

$$g(u) = u^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d} h(x) (h(x + (y - y')/u) - h(x)) dx \mu(dy) \mu(dy').$$

Then we use the relevant property of $h \in \mathcal{H}_\beta$ for this range of the parameter β to obtain an $\alpha \in (d - 1, \beta)$, such that the bounds in (15) and (16) are preserved. In parallel with the previous case it remains to integrate over u to obtain the covariance functional, which now yields a self-similarity index H in the range $0 < H < 1/2$.

To prove part ii) of the theorem we begin with the case $d \geq 2$, take β and m as specified, and consider the function

$$g_m(u) = \int_{\mathbb{R}^d} \langle (-\Delta)^{-m/2} \mu, \tau_{x,u} h \rangle^2 dx, \quad u > 0,$$

for $h \in \mathcal{H}_\beta$ and $(-\Delta)^{-m/2} \mu \in \mathcal{M}^\beta$. Using the notation

$$V_h(y) = \int_{\mathbb{R}^d} h(x) h(x+y) dx, \quad y \in \mathbb{R}^d,$$

we have

$$g_m(u) = u^d \int_{\mathbb{R}^d \times \mathbb{R}^d} V_h((y-y')/u) (-\Delta)^{-m/2} \mu(dy) (-\Delta)^{-m/2} \mu(dy').$$

By using $h \in \mathcal{H}_\beta$ and Hölder's inequality as in the proof of part 1), we find that g_m satisfies relation (16) for some α with $\beta < \alpha < 2d$, which implies that the covariance functional

$$C(\mu, \mu) = \int_0^\infty g_m(u) u^{-\beta-1} du = \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle (-\Delta)^{-m/2} \mu, \tau_z h \rangle^2 \nu(dz)$$

is finite. Moreover, by a change of variable and relation (3),

$$g_m(u) = u^d \int_{\mathbb{R}^d} V_h(w/u) \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\mu(dy) \mu(dy')}{|y-y'+w|^{d-2m}} dw.$$

Thus,

$$C(\mu, \mu) = \int_{\mathbb{R}^d} \frac{1}{|w|^{\beta-d}} \mathcal{K}_h\left(\frac{w}{|w|}\right) \frac{\mu(dy) \mu(dy')}{|y-y'+w|^{d-2m}} dw,$$

where

$$\mathcal{K}_h(e) = \text{const} \int_0^\infty u^{d-\beta-1} V_h(e/u) du, \quad e \in \mathbb{S}^{d-1},$$

is a finite function on the unit sphere. Clearly,

$$C(\mu_c, \mu_c) = c^{2H'} C(\mu, \mu), \quad H' = \frac{d-\beta+2m}{2}.$$

For $d = 1$, the arguments are parallel and lead to the representation

$$C(\mu, \mu) = \int_{\mathbb{R} \times \mathbb{R}} \int_0^\infty u^{-\beta} V_h\left(\frac{y-y'}{|y-y'|} \frac{1}{u}\right) du \frac{\mu([y, \infty)) \mu([y', \infty))}{|y-y'|^{\beta-1}} dy dy',$$

which scales with self-similarity index of order $(3-\beta)/2 \in (1/2, 1)$. \square

Theorem 3. Let $M_d(dx)$ be Gaussian white noise on \mathbb{R}^d with control measure dx . For $\beta \in (d-1, d) \cup (d, 2d)$, put $H = (d-\beta)/2$ and let h_β and h_β^+ be functions defined by

$$h_\beta(x) = |x|^{-\beta/2}, \quad x \in \mathbb{R}^d, \quad h_\beta^+(x) = x_+^{-\beta/2}, \quad x \in \mathbb{R}, \quad x_+ = x \vee 0.$$

i) The Gaussian random field

$$\mu \mapsto X(\mu) = \int_{\mathbb{R}^d} \langle \mu, \tau_x h_\beta \rangle M_d(dx), \quad \mu \in \widetilde{\mathcal{M}}_\beta,$$

is H -selfsimilar with covariance

$$\text{Cov}(X(\mu), X(\mu')) = \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y-y'|^{2H} \mu(dy) \mu'(dy').$$

For $d - 1 < \beta < d$ this may be written

$$\text{Cov}(X(\mu), X(\mu')) = C_+ \int_{\mathbb{R}^d \times \mathbb{R}^d} (|y|^{2H} + |y'|^{2H} - |y - y'|^{2H}) \mu(dy) \mu'(dy')$$

with a positive constant C_+ .

ii) Restricting to $d \geq 2$ and $d < \beta < 2d$, let m be a real number such that

$$0 < 2m < d, \quad 0 < d - \beta + 2m < 2.$$

For μ such that $(-\Delta)^{-m/2} \mu \in \mathcal{M}^\beta$ we have

$$\begin{aligned} \mu &\mapsto \int_{\mathbb{R}^d} \langle (-\Delta)^{-m/2} \mu, \tau_x h_\beta \rangle M_d(dx) \\ &= \text{const} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^{H' - d/2} \mu(dy) M_d(dx), \end{aligned}$$

and this map defines a selfsimilar Gaussian random field with self-similarity index $H' = (d - \beta)/2 + m \in (0, 1)$. For the case $d = 1$ and $1 < \beta < 2$, the random field

$$\mu \mapsto \int_{\mathbb{R}} \langle (-\Delta)^{-1/2} \mu, \tau_x h_\beta^+ \rangle M_1(dx) = \text{const} \int_{\mathbb{R}} \int_{\mathbb{R}} (y - x)_+^{H' - 1/2} \mu(dy) M_1(dx),$$

is H' -selfsimilar with $H' = (3 - \beta)/2 \in (1/2, 1)$. Also,

$$(17) \quad \mu \mapsto \int_{\mathbb{R}} (-\Delta)^{-1/2} \mu(x) M_1(dx) = \int_{\mathbb{R}} \int_x^\infty \mu(dy) M_1(dx)$$

is $1/2$ -selfsimilar.

Proof. To prove i) we need to establish that

$$\text{Cov}(X(\mu), X(\mu)) = \int_{\mathbb{R}^d} \langle \mu, h_\beta \rangle^2 dx$$

has the required properties. Indeed, there is a constant c_β such that

$$\int_{\mathbb{R}^d} \langle \mu, h_\beta \rangle^2 dx = c_\beta \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{d-\beta} \mu(dy) \mu(dy').$$

Here,

$$c_\beta = \int_{\mathbb{R}^d} h_\beta(x) h_\beta(x + e) dx < \infty,$$

some $e \in \mathbb{S}^{d-1}$, for the case $d < \beta < 2d$, and, using $\int \mu(dy) = 0$,

$$c_\beta = \int_{\mathbb{R}^d} h_\beta(x) (h_\beta(x + e) - h_\beta(x)) dx < \infty$$

for the case $d - 1 < \beta < d$. To prove ii) for $d \geq 2$, $d < \beta < 2d$, and m as specified, we have by (3),

$$\begin{aligned}
& \int_{\mathbb{R}^d} \langle (-\Delta)^{-m/2} \mu, \tau_x h_\beta \rangle M_d(dx) \\
&= \text{const} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\mu(dy)}{|y - w|^{d-m}} \frac{dw}{|w - x|^{\beta/2}} M_d(dx) \\
&= \text{const} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(dy)}{|y - x|^{\beta/2-m}} M_d(dx) \\
&= \text{const} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^{H' - d/2} \mu(dy) M_d(dx),
\end{aligned}$$

and we can check as before that the covariance is finite under the given assumptions. The proof for the one-dimensional case, which uses $m = 1$, is analogous. \square

Random cylinder Gaussian fields. The purpose of this subsection is to show that Brownian sheets models are naturally included in the general framework of selfsimilar random fields, and that they emerge from expanding the white noise construction in Theorem 2 based on random balls to one based on random cylinders. In the interest of not burdening our main result Theorem 2 with additional notation and variations we have chosen to present these results in a separate subsection and in a less formal manner.

We define random cylinder white noise on the product space $\mathbb{R}^d \times \mathbb{R}_+^p$, $1 \leq p \leq d$, equipped with a control measure that allows us to think of the noise as Gaussian fluctuations of overlapping random cylinders. The special case $p = 1$ is the random balls white noise. For a given spatial integer dimension d we consider an arbitrary partition $d^\pi = (d_1, \dots, d_p)$ of d , $d = \sum_{i=1}^p d_i$. Any point $x \in \mathbb{R}^d = \prod_{i=1}^p \mathbb{R}^{d_i}$ has the representation $x^\pi = (x^1, \dots, x^p)$, where $x^i \in \mathbb{R}^{d_i}$ for $1 \leq i \leq p$. Given a set of parameters $\tilde{\beta} = (\beta_1, \dots, \beta_p)$ such that either $d_i < \beta_i < 2d_i$, $1 \leq i \leq p$, or $d_i - 1 < \beta_i < d_i$, $1 \leq i \leq p$, we define a measure $\tilde{\nu}_\beta(du)$ on $\mathbb{R}_+^p = [0, \infty)^p$ by

$$(18) \quad \tilde{\nu}_\beta(du) = \prod_{i=1}^p u_i^{-\beta_i-1} du_i.$$

The scaling relation

$$\tilde{\nu}_\beta(c du) = c^{-\beta} \tilde{\nu}_\beta(du), \quad c > 0, \quad \beta = \sum_{i=1}^p \beta_i,$$

holds. Let $M_\beta(dz)$ be a Gaussian measure on $\mathbb{R}^d \times \mathbb{R}_+^p$ defined by the intensity measure $\nu(dz) = dx \tilde{\nu}_\beta(du)$. With each point $z = (x, u)$ we associate a shift and scale operator $\tau_z h : \mathbb{R}^d \mapsto \mathbb{R}^{d+p}$ acting on functions $h \in L_2(\mathbb{R}^d)$, by

$$\tau_z h(y) = h((y^1 - x^1)/u_1, \dots, (y^p - x^p)/u_p).$$

In particular, letting h be the indicator function of the partition unit ball $C(0, 1) = \{y^\pi \in \mathbb{R}^d : |y^i|_{d_i} \leq 1, 1 \leq i \leq p\}$, where $|\cdot|_k$ is the euclidean norm in \mathbb{R}^k , it follows that $\tau_z h$ with $z = (x, u)$ is the indicator function of the random cylinder $C(x, u)$ with center point $x \in \mathbb{R}^d$ and partition radius u , that is

$$C(x, u) = \{y^\pi \in \mathbb{R}^d : |y^i - x^i|_{d_i} \leq u_i, 1 \leq i \leq p\}, \quad x^\pi \in \mathbb{R}^d, u \in \mathbb{R}_+^p.$$

The map τ_z has the invariance property

$$\tau_z h(cy) = \tau_{z/c} h(y), \quad y \in \mathbb{R}^d, z \in \mathbb{R}^{d+p}, \quad c > 0.$$

In analogy to the shot noise model we define the cylinder random field by

$$X_h(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+^p} \langle \mu, \tau_z h \rangle M_\beta(dz).$$

By proper modifications of the arguments given in the previous sections one can show that the generalized random field $\mu \mapsto X(\mu)$ is well-defined for a suitably restricted class of measures. For simplicity we focus on the simplest case $h(y) = \mathbb{I}_{\{|y| \leq 1\}}$ in the rest of this section, and hence consider

$$X(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+^p} \mu(C(x, u)) M_\beta(dz).$$

The covariance functional is

$$\begin{aligned} C(\mu, \mu') &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mu(dy) \mu'(dy') \int_{\mathbb{R}_+^p} |C(y, u) \cap C(y', u)| \prod_{i=1}^p u_i^{-\beta_i-1} du_i \\ &= \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mu(dy) \mu'(dy') \prod_{i=1}^p |y^i - y'^i|_{d_i}^{d_i - \beta_i}. \end{aligned}$$

Put

$$H = \sum_{i=1}^p (d_i - \beta_i) = d - \beta \in (-d/2, 0) \cap (0, 1/2).$$

Then $C(\mu_c, \mu'_c) = c^{2H} C(\mu, \mu')$ and it follows that the cylinder random field is selfsimilar with index H . To recognize this model as an instance of Theorem 1, let \mathcal{K} be the function on \mathbb{S}^{d-1} defined such that if $e \in \mathbb{S}^{d-1}$ has decomposition $e^\pi = (e^1, \dots, e^p)$, then

$$\mathcal{K}(e) = \prod_{i=1}^p |e^i|_{d_i}^{d_i - \beta_i}, \quad e = e^\pi \in \mathbb{S}^{d-1}.$$

Then

$$C(\mu, \mu') = \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mu(dy) \mu'(dy') \mathcal{K}\left(\frac{y - y'}{|y - y'|_d}\right) |y - y'|_d^{d-\beta}.$$

4. EXTRACTING GAUSSIAN PROCESSES FROM THE RANDOM FIELDS

The main tool for extracting random processes indexed by points on the real line or points in Euclidean space, from abstract random fields $X(\mu)$ indexed by measures μ , will be to evaluate the random fields using specifically chosen families of measures, such as $\mu_t = \delta_t - \delta_0$, $0, t \in \mathbb{R}^d$, $d \geq 1$.

Fractional Brownian motion. Fractional Brownian motion on \mathbb{R}^d is a parametrized class of pointwise defined, centered Gaussian random fields $B_H(t)$, $t \in \mathbb{R}^d$, defined by the covariance functional

$$\text{Cov}(B_H(s), B_H(t)) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \quad s, t \in \mathbb{R}^d,$$

where the parameter H , called the Hurst index, ranges over $0 < H < 1$ and is the self-similarity index in the sense of $\{B_H(ct)\} \stackrel{d}{=} \{c^H B_H(t)\}$, $c > 0$. The case $H = 1/2$ is known as Lévy Brownian motion. See [6] and [17] for the general theory of such processes.

Next we show how to obtain B_H from the selfsimilar Gaussian random fields constructed in Theorem 2. In part i) of the theorem we take β such that $d-1 < \beta < d$ and a rotationally symmetric function h on \mathbb{R}^d such that $h \in \mathcal{H}_\beta$. Then $\mu = \delta_t - \delta_0 \in \widetilde{\mathcal{M}}_\beta$, and the map

$$t \mapsto X_h(\delta_t - \delta_0) = \int_{\mathbb{R}^d \times \mathbb{R}_+} (h((t-x)/u) - h(-x/u)) M_\beta(dx, du),$$

defines a zero mean Gaussian random field with covariance function

$$C(s, t) = \text{Cov}(X_h(\delta_s - \delta_0), X_h(\delta_t - \delta_0))$$

given by

$$\begin{aligned} C(s, t) &= \text{const} \int_{\mathbb{R}^d \times \mathbb{R}_+} |y - y'|^{d-\beta} (\delta_s - \delta_0)(dy) (\delta_t - \delta_0)(dy') \\ &= c_h (|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \end{aligned}$$

which is a multiple of fractional Brownian motion with Hurst index $H \in (0, 1/2)$. In particular, with $h(y) = \mathbb{I}_{\{|y| \leq 1\}}$ we have

$$X_h(\delta_t - \delta_0) = \int_{\mathbb{R}^d \times \mathbb{R}_+} (\delta_t(B(x, u)) - \delta_0(B(x, u))) M_\beta(dx, du).$$

This representation of $B_H(t)$ for the case $0 < H < 1/2$ is discussed in [3] and may be recognized as a so called $(2, H)$ -Takenaka field $B_H(t) = M_\beta(V_t)$, where

$$\begin{aligned} V_t &= \{\text{all spheres separating } 0 \text{ and } t\} \\ &= \{(x, r) : |x| \leq r\} \Delta \{(x, r) : |x-t| \leq r\}, \end{aligned}$$

where Δ denotes the symmetric difference of two sets in \mathbb{R}^d , see [17]. Next, in part ii) of Theorem 2 we consider $d \geq 2$ and take β and m such that $d < \beta < 2d$, $0 < d - \beta + 2m < 2$ and $0 < 2m < d$, and pick $h \in \mathcal{H}_\beta$ again rotationally symmetric. To show that the measure

$$(-\Delta)^{-m/2}(\delta_t - \delta_0)(dy) = C_{m,d} \left(\frac{1}{|t-y|^{d-m}} - \frac{1}{|y|^{d-m}} \right) dy$$

belongs to \mathcal{M}_β , we observe

$$\begin{aligned} &(-\Delta)^{-m/2}(\delta_t - \delta_0) * (-\Delta)^{-m/2}(\delta_t - \delta_0)(dy) \\ &= C_{2m,d} \left(\frac{2}{|y|^{d-2m}} - \frac{1}{|t+y|^{d-2m}} - \frac{1}{|t-y|^{d-2m}} \right) dy \end{aligned}$$

and

$$\int_{\mathbb{R}^d} \frac{1}{|y|^{\beta-d}} \left| \frac{2}{|y|^{d-2m}} - \frac{1}{|t+y|^{d-2m}} - \frac{1}{|t-y|^{d-2m}} \right| dy < \infty.$$

Thus, under the stated assumptions,

$$(19) \quad t \mapsto \int_{\mathbb{R}^d \times \mathbb{R}_+} \langle (-\Delta)^{-m/2}(\delta_t - \delta_0), \tau_z h \rangle M_\beta(dz), \quad t \in \mathbb{R}^d,$$

is a multiple of fractional Brownian motion with Hurst index $H' = H + m \in (0, 1)$. As an explicit example, with $h(y) = \mathbb{I}_{\{|y| \leq 1\}}$ we can find a constant C , such that

$$B_{H'}(t) \stackrel{d}{=} C \int_{\mathbb{R}^d \times \mathbb{R}_+} \int_{B(x, u)} \left(\frac{1}{|t-y|^{d-m}} - \frac{1}{|y|^{d-m}} \right) dy M(dx, du),$$

where $M(dx, du)$ is Gaussian white noise on $\mathbb{R}^d \times \mathbb{R}_+$ with control measure $\nu(dx, du) = dx u^{2H'-d-1-2m} du$. The special choice of parameters $d - \beta + 2m = 1$ with $1 < 2m < d$,

for which $H' = 1/2$, shows that Lévy Brownian motion is covered by this construction. In particular, letting $M(dx, du)$ have control measure $\nu(dx, du) = dx u^{-d-2m} du$,

$$B_{1/2}(t) \stackrel{d}{=} C \int_{\mathbb{R}^d \times \mathbb{R}_+} \int_{B(x,u)} \left(\frac{1}{|t-y|^{d-m}} - \frac{1}{|y|^{d-m}} \right) dy M(dx, du).$$

Our corresponding result for dimension $d = 1$ is less general in the sense that $1 < \beta < 2$, $m = 1$ and $H' = (3 - \beta)/2 \in (1/2, 1)$. Random balls representations for the one-dimensional model with this range of Hurst index have been studied earlier, see e.g. [13], [14]. Now

$$(20) \quad (-\Delta)^{-1/2}(\delta_t - \delta_0) = \int_x^\infty (\delta_t - \delta_0)(dy) = 1_{[0,t]}(x).$$

Hence, letting $M(dx, du)$ be a Gaussian measure on $\mathbb{R} \times \mathbb{R}_+$ with control measure $\nu(dx, du) = dx u^{2H'-4} du$,

$$B_{H'}(t) \stackrel{d}{=} C \int_{\mathbb{R} \times \mathbb{R}_+} \int_0^t h((y-x)/u) dy M(dx, du).$$

We conclude this subsection by comparing the representations of fractional Brownian motion obtained above with those we get by taking $\mu = \delta_t - \delta_0$ in Theorem 3. For $d-1 < \beta < d$ this choice of μ in Theorem 3 i), generates the map

$$\begin{aligned} t \mapsto \int_{\mathbb{R}^d} \langle \delta_t - \delta_0, h_\beta \rangle M_d(dx) &= \int_{\mathbb{R}^d} (h_\beta(t-x) - h_\beta(-x)) M_d(dx) \\ &= \int_{\mathbb{R}^d} (|t-x|^{H-d/2} - |x|^{H-d/2}) M_d(dx) \end{aligned}$$

for $H \in (0, 1/2)$, which we recognize as the so called well-balanced representation of fractional Brownian motion. By replacing h_β with h_β^+ for the case $d = 1$, we obtain the classical Mandelbrot and van Ness representation

$$(21) \quad B_H(t) \stackrel{d}{=} \int_{\mathbb{R}^d} ((t-x)_+^{H-1/2} - (-x)_+^{H-1/2}) M(dx), \quad t \geq 0.$$

for $0 < H < 1/2$. Similarly, Theorem 3 ii) with $\mu = \delta_t - \delta_0$ also yields a pointwise well-defined random process on \mathbb{R}^d given by

$$\begin{aligned} t \mapsto \int_{\mathbb{R}^d} \left(\frac{1}{|t-y|^{d-m}} - \frac{1}{|y|^{d-m}} \right) \frac{1}{|y-x|^{\beta/2}} dy M_d(dx) \\ = \text{const} \int_{\mathbb{R}^d} (|t-y|^{H'-d/2} - |y|^{H'-d/2}) M_d(dx), \end{aligned}$$

which again is the well-balanced representation of fractional Brownian motion with Hurst index $H' \in (0, 1)$. Finally, the case $d = 1$ in Theorem 3 applies the one-sided pulse function $h(x) = x_+^{-\beta/2}$ on the real line, and hence extends the Mandelbrot and van Ness representation (21) to the entire range of Hurst index $0 < H < 1$. In particular, by (17) and (20),

$$(22) \quad W_t = \int_{\mathbb{R}} (-\Delta)^{-1/2}(\delta_t - \delta_0)(x) M_1(dx), \quad t \geq 0,$$

is Brownian motion.

Examples of the cylinder model. a) $\nu_1 = d$: This is the heavy-tailed, one-parameter random balls model with $r = 1$ and $d < \beta < 2d$, for which

$$C(\mu, \eta) = V_d \int_{\mathbb{R}^d \times \mathbb{R}^d} \mu(dy) \eta(dy') |y - y'|_d^{d-\beta}.$$

b) The case $r = d$, $\nu_1 = \dots = \nu_d = 1$ gives a non-symmetric random sheets model with d parameters, $1 < \beta_i < 2$, such that

$$C(\mu, \eta) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mu(dy) \eta(dy') \prod_{i=1}^d C_i |y_i - y'_i|_1^{1-\beta_i}.$$

Take product measures $\mu(dy) = \prod_{i=1}^d \mu_i(dy_i)$ and $\eta(dy) = \prod_{i=1}^d \eta_i(dy_i)$ to obtain

$$C(\mu, \eta) = \prod_{i=1}^d C_i \int_{\mathbb{R} \times \mathbb{R}} \mu_i(dy_i) \eta_i(dy'_i) |y_i - y'_i|_1^{1-\beta_i}.$$

In particular, $\mu_i(A) = \int_0^{t_i} \mathbb{I}_A(y) dy$ and $\eta_i(A) = \int_0^{s_i} \mathbb{I}_A(y) dy$, yields

$$C(\mu, \eta) = \prod_{i=1}^d C_i \int_0^{t_i} \int_0^{s_i} \frac{dy_i dy'_i}{|y_i - y'_i|_1^{\beta_i-1}} = \prod_{i=1}^d C'_i \left(|t_i|_1^{3-\beta_i} + |s_i|_1^{3-\beta_i} - |t_i - s_i|_1^{3-\beta_i} \right).$$

The Gaussian free field. The choice of parameters $H = -d/2 + 1$ and $r = 1$ for the isotropic case in Theorem 1 is sometimes referred to as the Gaussian free field. By (7), the case $d = 1$, $H = 1/2$ has covariance functional

$$C(\mu, \mu') = \text{const} \int_{\mathbb{R} \times \mathbb{R}} |x - y| \mu(dx) \mu'(dy), \quad \mu, \mu' \in \widetilde{\mathcal{M}}.$$

With $\mu = \delta_t - \delta_0$ and $\mu' = \delta_s - \delta_0$ this gives Brownian motion:

$$\text{const} \int_{\mathbb{R} \times \mathbb{R}} |x - y| (\delta_t - \delta_0)(dx) (\delta_s - \delta_0)(dy) = \text{const} \min(s, t).$$

For $d \geq 3$ we have $-d/2 < H < 0$, which is a case covered by Theorem 2 i) with $\beta = 2(d-1)$ and $h \in \mathcal{H}_\beta$ rotationally symmetric. The control measure of the driving Gaussian random balls white noise is $\nu(dx, du) = dx u^{-2d+1} du$.

The remaining case $d = 2$, $H = 0$ is not included in Theorem 1. However, white noise $M_2(dx)$ for $d = 2$ has $H = -1$ and hence it is natural to consider for the free field

$$X(\varphi) = M_2((-\Delta)^{-1/2} \varphi), \quad \varphi \in \mathcal{S}_1,$$

with covariance functional

$$\begin{aligned} C(\varphi, \psi) &= \int_{\mathbb{R}^2} (-\Delta)^{-1/2} \varphi(x) (-\Delta)^{-1/2} \psi(x) dx \\ &= \int_{\mathbb{R}^2} \varphi(x) (-\Delta)^{-1} \psi(x) dx, \end{aligned}$$

where

$$(-\Delta)^{-1} \psi(x) = \int_{\mathbb{R}^2} \psi(y) G(x - y) dy$$

and $G(x)$ is the Green's function of Brownian motion in $d = 2$. Thus,

$$C(\varphi, \psi) = - \int_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x) \psi(y) \log(|x - y|) dx dy,$$

which is known as de Wijs random field, see [2] for a background.

Generalized Gaussian bridges. For a continuous Gaussian process $X = (X_t)_{t \in [0, T]}$ and a signed finite Borel measure a on an interval $[0, T]$ of the real line, denote by $X^{(a)} = (X_t^{(a)})_{t \in [0, T]}$ the process X conditioned on the event that $a(X) = \int_0^T X_t a(dt) = 0$. Such generalized Gaussian bridges are studied in [1], [10], and [18]. It is shown that the conditioned process $X^{(a)}$ admits a representation of the form

$$X_t^{(a)} = X_t - f_{(a)}(t)a(X), \quad 0 \leq t \leq T,$$

for a suitable continuous function $f_{(a)} : [0, T] \rightarrow \mathbb{R}$. For example, Brownian motion $W = (W_t)_{t \in [0, T]}$ conditioned on $W_1 = 0$ is obtained from $a = \delta_1$ and yields the Brownian bridge $B = (B_t)_{t \in [0, T]}$ with representation $B_t = W_t - tW_1$, $0 \leq t \leq 1$.

To obtain the Brownian bridge B in our setting of extracting Gaussian processes from random fields, we apply relation (17) of Theorem 3 with the special choice

$$(23) \quad \mu = \delta_t - \delta_0 - t(\delta_1 - \delta_0) \in \mathcal{M}_2.$$

and use the linearity of the mapping $\mu \mapsto M((-\Delta)^{-1/2}\mu)$ to see

$$(24) \quad B_t = M_1((-\Delta)^{-1/2}(\delta_t - (1-t)\delta_0 - t\delta_1)), \quad 0 \leq t \leq 1.$$

This observation generalizes as follows: consider $X_t = M((-\Delta)^{-1/2}\mu_t)$, for an unspecified family of measures $\mu_t \in \mathcal{M}_1$, $0 \leq t \leq T$, and assume that $(X_t)_{t \in [0, T]}$ is continuous on $[0, T]$ almost surely. Let a be a signed finite Borel measure on $[0, T]$ and for suitable continuous $f^{(a)} : [0, T] \rightarrow \mathbb{R}$, define

$$\mu_t^{(a)}(A) = \mu_t(A) - f^{(a)}(t) \int_0^T \mu_s(A) a(ds),$$

for Borel sets $A \subset [0, 1]$. Then the conditioned process $X^{(a)}$ has the representation

$$X_t^{(a)} = X_t - f^{(a)}(t)a(X) = M_1((-\Delta)^{-1/2}\mu_t^{(a)}).$$

Indeed,

$$\begin{aligned} X_t^{(a)} &= \int_{\mathbb{R}} \int_x^\infty \mu_t(dy) M_1(dx) - f^{(a)}(t) \int_0^T \int_{\mathbb{R}} \int_x^\infty \mu_s(dy) M_1(dx) a(ds) \\ &= \int_{\mathbb{R}} \int_x^\infty \left[\mu_t - f^{(a)}(t) \int_0^T \mu_s a(ds) \right] (dy) M_1(dx). \end{aligned}$$

As a concrete example we consider the Brownian bridge B with representation (24) conditioned to have vanishing Lebesgue measure on $[0, 1]$, i.e., $a(dt) = dt$. In [10], the resulting conditioned process $B^{(a)}$ is called the zero area Brownian bridge, and is shown to satisfy

$$B_t^{(a)} = B_t - 6t(1-t)a(B).$$

Here, we recover this relation as $B^{(a)}(t) = M_1((-\Delta)^{-1/2}\mu_t^{(a)})$ with

$$\begin{aligned} \mu_t^{(a)}(A) &= \mu_t(A) - 6t(1-t) \int_0^T \mu_s(A) ds \\ &= \delta_t(A) - \delta_0(A)(1-4t+3t^2) + \delta_1(A)(2t-3t^2) + |A|6(t^2-t) \end{aligned}$$

for Borel sets $A \subset [0, 1]$. One can check that $\mu_t^{(a)} \in \mathcal{M}_3$.

Volterra processes. The representation of the Brownian bridge in (24) involves the measure densities

$$(-\Delta)^{-1/2}(\delta_t - (1-t)\delta_0 - t\delta_1)(x) = (1-t)\mathbb{I}_{[0,t]}(x) - t\mathbb{I}_{(t,1]}(x), \quad t \in [0,1],$$

supported on $[0,1]$. However, the Brownian bridge also admits a representation as a Volterra process of the form

$$B_t = \int_0^t \frac{1-t}{1-x} M_1(dx).$$

The question arises if we are able to define measures $(\mu_t)_{t \in [0,1]}$ on \mathbb{R} such that the support of $(-\Delta)^{-1/2}\mu_t$ is a subset of $[0,t]$ and $B_t = M_1((-\Delta)^{-1/2}\mu_t)$.

Let $X = (X_t)_{t \in [0,\infty)}$ be a Gaussian Volterra process with

$$X_t = \int_0^t K(t,x) M_1(dx)$$

and assume that the kernel K is defined on $\mathbb{R} \times \mathbb{R}$ with $K(t,x) = K(t,x)\mathbb{I}(0 < x \leq t)$. Moreover, assume that $K(t,\cdot)$ is continuous from the left and has limits from the right and has finite total variation. Then the measures $(\mu_t)_{t \in \mathbb{R}}$ defined by $\mu_t((-\infty, x)) = -K(t,x)$ are admissible in the sense $\mu_t \in \mathcal{M}_1$, and $(-\Delta)^{-1/2}\mu_t$ is supported on $[0,t]$. If $K(t,x)$ is differentiable for $0 < x < t$ then

$$\mu_t(dx) = K(t,t)\delta_t(dx) - K(t,0+)\delta_0(dx) - \frac{\partial}{\partial x}K(t,x)\mathbb{I}_{(0,t]}(x)dx.$$

By defining μ_t in this manner it follows that $X_t = M_1((-\Delta)^{-1/2}\mu_t)$. In fact,

$$M_1((-\Delta)^{-1/2}\mu_t) = \int_0^\infty \int_x^\infty \mu_t(dy) M_1(dx) = \int_0^\infty \mu_t([x,\infty)) M_1(dx)$$

and so, since $\mu_t(\mathbb{R}) = 0$,

$$\begin{aligned} M_1((-\Delta)^{-1/2}\mu_t) &= \int_0^\infty -\mu_t((-\infty, x)) M_1(dx) \\ &= \int_0^\infty K(t,x) M_1(dx) = \int_0^t K(t,x) M_1(dx). \end{aligned}$$

We give some examples. Of course, the simplest example is Brownian motion, where $K(t,x) = \mathbb{I}_{(0,t]}(x)$ and $\mu_t(dx) = \delta_t(dx) - \delta_0(dx)$. For $\alpha \geq 0$ and $0 \leq t < 1$ define

$$X_t^{(\alpha)} = \int_0^t \left(\frac{1-t}{1-x} \right)^\alpha M_1(dx).$$

For $\alpha = 0$ we get the Brownian motion and for $\alpha = 1$ we get the usual Brownian bridge. The integration kernel $K(t,x)$ is such that, for $0 < x < t$,

$$K(t,x) = (1-t)^\alpha(1-x)^{-\alpha}\mathbb{I}_{(0,t]}(x), \quad \frac{\partial}{\partial x}K(t,x) = \alpha(1-t)^\alpha(1-x)^{-\alpha-1}.$$

Hence the measures $(\mu_t)_{t \in [0,1]}$ become

$$\mu_t(dx) = \delta_t(dx) - (1-t)^\alpha\delta_0(dx) - \alpha \frac{(1-t)^\alpha}{(1-x)^{1+\alpha}} \mathbb{I}_{(0,t]}(x) dx.$$

A further example is the centered Ornstein-Uhlenbeck process with stability parameter $\alpha > 0$ and diffusion parameter $\sigma > 0$, given by

$$X_t^{(\alpha, \sigma)} = \int_0^t \sigma e^{\alpha(x-t)} dW_x.$$

The kernel is $K(t, x) = \sigma e^{\alpha(x-t)} \mathbb{I}_{(0, t]}(x)$ and thus $\frac{\partial}{\partial x} K(t, x) = \alpha \sigma e^{\alpha(x-t)}$, $0 < x < t$. Therefore

$$\mu_t(dx) = \sigma \delta_t(dx) - \sigma e^{-\alpha t} \delta_0(dx) - \alpha \sigma e^{\alpha(x-t)} \mathbb{I}_{(0, t]}(x) dx.$$

Fractional bridges. For $0 < H < 1$, let B_H be a standard fractional Brownian motion on the real line and let a_t be the function on the unit interval defined by

$$a_t^H = \frac{1 + t^{2H} - (1-t)^{2H}}{2}, \quad 0 \leq t \leq 1.$$

It is known that the fractional Brownian bridge process obtained by pinning B_H to zero at time $t = 1$ is equal in distribution to $Y_t = B_H(t) - a_t^H B_H(1)$, see [9], [10].

To obtain the fractional Brownian bridge in this work we apply Theorem 3 i) for $d = 1$ and $0 < \beta < 1$ to get

$$Y_t \stackrel{d}{=} \text{const} X(\delta_t - \delta_0 - a_t^H(\delta_1 - \delta_0)), \quad 0 < H < 1/2.$$

Similarly, by Theorem 3 ii) for $d = 1$, $1 < \beta < 2$, and $1/2 < H < 1$,

$$Y_t \stackrel{d}{=} \text{const} \int_{\mathbb{R}} \left((t-x)_+^{H-1/2} - (1-a_t^H)(-x)_+^{H-1/2} - a_t^H(1-x)_+^{H-1/2} \right) M_1(dx).$$

Membranes by soft boundary thinning. In this subsection we discuss briefly an approach of extending the method for extracting Gaussian bridge processes in one dimension to a method for extracting Gaussian membranes in higher dimensions.

We start again with the representation $B_t = M_1((-\Delta)^{-1/2} \mu_t)$ in (24) of the Brownian bridge on $[0, 1]$. Here $\mu_t = \delta_t - \omega_t$, where the measure $\omega_t = (1-t)\delta_0 - t\delta_1$ is the harmonic measure on the set $\{0, 1\}$ (the start and end point of the bridge) of a Brownian motion starting in t . This observation leads us to defining a class of Gaussian membranes as follows. Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the Poincaré cone condition (see Definition 3.10 in [16]). For $t \in D$, let ω_t denote the harmonic measure

$$\omega_t(A) = \mathbb{P}(W(\tau) \in A), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $\{W(x), x \in \mathbb{R}^d\}$ is d -dimensional Brownian motion with $W(0) = t$ and τ is the stopping time $\tau = \inf\{s \geq 0 : W(s) \in \partial D\}$, and let μ_t in this subsection denote the signed measure

$$\mu_t = \delta_t - \omega_t, \quad t \in D.$$

Given a continuous function $f : \partial D \rightarrow \mathbb{R}$, a solution to the Dirichlet problem with boundary value f is a function $u : \bar{D} \rightarrow \mathbb{R}$ which is harmonic in D and satisfies $u(t) = f(t)$ for $t \in \partial D$. It can be shown that the unique solution is

$$(25) \quad u(t) = \int_{\mathbb{R}^d} f(y) \omega_t(dy), \quad t \in \bar{D},$$

see [16], Corollary 3.40.

We have $\int_{\mathbb{R}^d} \mu_t(dy) = 0$ and, since the function $u(t) = t$ is harmonic on \mathbb{R}^d ,

$$\int_{\mathbb{R}^d} y \mu_t(dy) = t - \int_{\partial D} y \omega_t(dy) = 0.$$

Thus, $\mu_t \in \mathcal{M}_2$. Considering now the Gaussian selfsimilar random fields $\mu \mapsto X(\mu)$ in Theorem 3 i), or Theorem 2 i), with self-similarity index $H \in (0, 1/2)$, we can introduce for $d \geq 1$ a collection $(X_t)_{t \in \bar{D}}$ of zero mean Gaussian random variables defined by $t \mapsto X_t = X(\mu_t)$, with finite covariance

$$(26) \quad \mathbb{E}X_s X_t = \text{const} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{2H} \mu_s(dy) \mu_t(dy'), \quad s, t \in D.$$

To construct a Brownian membrane on D vanishing on ∂D , we apply Theorem 3 ii) with $H = 1/2$ and put

$$X_t = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu_t(dy)}{|y - x|^{(d-1)/2}} M_d(dx), \quad t \in \mathbb{R}^d.$$

The variance now is

$$(27) \quad \mathbb{E}X_t^2 = \int_{\mathbb{R}^d} \left(\frac{1}{|t - x|^{(d-1)/2}} - \int_{\partial D} \frac{\omega_t(dy)}{|y - x|^{(d-1)/2}} \right)^2 dx,$$

assuming the integral exists. Here, we restrict to $H = 1/2$ but the same construction works for $H < 1$.

The following Proposition justifies the term membrane for this class of processes in the domain D which vanishes on the boundary ∂D .

Proposition 1. *The processes $(X_t)_{t \in \bar{D}}$ described above for $0 < H \leq 1/2$ are well-defined, and for $x \in \partial D$ we have $\lim_{t \rightarrow x} \mathbb{E}X_t^2 = 0$.*

Proof. First, we show that, for $x \in \partial D$, we have $\lim_{t \rightarrow x} \omega_t = \delta_x$ in the weak sense: Assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and bounded. By (25), the function

$$u(t) = \int_{\mathbb{R}^d} f(y) \omega_t(dy),$$

is continuous on \bar{D} with $u(t) = f(t)$ on ∂D , and thus

$$\lim_{t \rightarrow x} \int_{\mathbb{R}^d} f(y) \omega_t(dy) = \lim_{t \rightarrow x} u(t) = u(x) = f(x) = \int_{\mathbb{R}^d} f(y) \delta_x(dy).$$

Next, for $0 < H < 1/2$ by (26),

$$\mathbb{E}X_t^2 = \text{const} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{2H} \omega_t(dy) \omega_t(dy') - 2 \int_{\mathbb{R}^d} |t - y|^{2H} \omega_t(dy) \right).$$

From the first part of the proof it follows that $\omega_t \otimes \omega_t \rightarrow \delta_x \otimes \delta_x$ as $t \rightarrow x$ and thus

$$\begin{aligned} \lim_{t \rightarrow x} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{2H} \omega_t(dy) \omega_t(dy') \\ = \lim_{t \rightarrow x} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{2H} \delta_x(dy) \delta_x(dy') = 0. \end{aligned}$$

Moreover, we have $\delta_t \otimes \omega_t \rightarrow \delta_x \otimes \delta_x$ as $t \rightarrow x$. Hence,

$$\begin{aligned} \lim_{t \rightarrow x} \int_{\mathbb{R}^d} |t - y|^{2H} \omega_t(dy) &= \lim_{t \rightarrow x} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{2H} \omega_t(dy) \delta_t(dy') \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - y'|^{2H} \delta_x(dy) \delta_x(dy') = 0. \end{aligned}$$

Turning to the case $H = 1/2$, by rewriting (27),

$$\mathbb{E}X_t^2 = \int_{\partial D \times \partial D} F_t(y, y') \omega_t(dy) \omega_t(dy'),$$

where, using the short notation $\delta = (d-1)/2$,

$$F_t(y, y') = \int_{\mathbb{R}^d} \left(\frac{1}{|z|^\delta} - \frac{1}{|z+y-t|^\delta} - \frac{1}{|z+t-y'|^\delta} + \frac{1}{|z+y-y'|^\delta} \right) \frac{dz}{|z|^\delta}.$$

For any $t \in \mathbb{R}^d$, the function F_t is bounded on $\partial D \times \partial D$. Hence,

$$\mathbb{E}X_t^2 \rightarrow F_x(x, x) = 0, \quad t \rightarrow x \in \partial D.$$

□

5. GAUSSIAN MEMBRANES BY HARD BOUNDARY THINNING

In this section we consider another way to construct membranes on domains in \mathbb{R}^d . Again, let $D \subset \mathbb{R}^d$ be a bounded domain, and let $\beta < d$ be a real number. We modify the basic model (10) and consider

$$X(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \mu(B(x, u)) M_\beta^D(dx, du),$$

where M^D is the Gaussian random measure on $\mathbb{R}^d \times \mathbb{R}_+$ with control measure $\nu^D(dx, du) = u^{-\beta-1} \mathbb{I}(B(x, u) \subset D) dx du$. Hence in the random balls interpretation, the intensity measure is modified such that balls which do not fall entirely inside the domain D are discarded. This model is well-defined for any $\mu \in \mathcal{M}$ and $\beta < d$. Here, we will apply the extraction principle $t \mapsto X(\delta_t)$ and consider

$$(28) \quad W_\beta(t) = \int_{\mathbb{R}^d \times \mathbb{R}_+} \delta_t(B(x, u)) M_\beta^D(dx, du), \quad t \in \mathbb{R}^d.$$

Since D is bounded there is an $N > 0$ such that $\mathbb{I}(B(x, u) \subset D) = 0$ for all $u > N$ and $x \in D$. Hence, W_β is a Gaussian random field with finite covariance function

$$(29) \quad \begin{aligned} \mathbb{E}W_\beta(s)W_\beta(t) &= \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{I}(s, t \in B(x, u) \subset D) u^{-\beta-1} dx du \\ &\leq \int_0^N |B(s, u) \cap D| u^{-\beta-1} du < \infty, \end{aligned}$$

for all $s, t \in \mathbb{R}^d$. In particular, $W_\beta(s)$ and $W_\beta(t)$ are independent if and only if there is no $x \in \mathbb{R}^d$ and $u > 0$ such that $B(x, u)$ is a subset of D and covers both points $s \in \mathbb{R}^d$ and $t \in \mathbb{R}^d$. By the Lebesgue dominated convergence theorem, $\mathbb{E}(W_\beta(t)^2) \rightarrow 0$ as $t \rightarrow t_0 \in \partial D$, which again justifies the notion of a membrane.

The random fields W_β are not selfsimilar in the sense that $W(cs) \stackrel{d}{=} c^H W(s)$ for some H . But we will see that they are selfsimilar in the following local sense (see [8] for more details and background): A zero mean random field $W = (W(t))_{t \in E}$, $E \subset \mathbb{R}^d$ open, is said to be locally asymptotically selfsimilar with index H in the point $z \in E$, if H is the supremum of all $\gamma \geq 0$ such that

$$(30) \quad \varepsilon^{-\gamma} (W(z + \varepsilon s) - W(z)) \rightarrow 0$$

as $\varepsilon \rightarrow 0$ in the sense of finite dimensional distributions. Then the random field $T^z = (T^z(s))_{s \in \mathbb{R}^d}$ with

$$(31) \quad T^z(s) = \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon) (W(z + \varepsilon s) - W(z))$$

is called the tangent field at $z \in \mathbb{R}^d$, if τ is a suitable scaling function such that the limit exists in the sense of finite dimensional distributions and $T^z \neq 0$. The tangent field is selfsimilar with index H and uniquely determined modulo constants. By (30), $\tau(\varepsilon)\varepsilon^\gamma \rightarrow \infty$ as $\varepsilon \rightarrow 0$ for

all $\gamma < H$ and $\tau(\varepsilon)\varepsilon^\gamma \rightarrow 0$ as $\varepsilon \rightarrow 0$ for all $\gamma > H$. It is not necessarily the case, however, that $\tau(\varepsilon) \sim c\varepsilon^{-H}$, some $c > 0$.

Theorem 4. *The Gaussian membrane W_β is in every point $z \in D$ locally asymptotically selfsimilar with index*

$$H = \begin{cases} (d - \beta)/2, & d - 1 < \beta < d, \\ 1/2, & \beta \leq d - 1. \end{cases}$$

Moreover, the tangent field T^z in z is a multiple of fractional Brownian motion with Hurst index H . The scaling function τ is given by

$$(32) \quad \tau(\varepsilon) = \begin{cases} \varepsilon^{(\beta-d)/2}, & d - 1 < \beta < d, \\ (-\varepsilon \ln(\varepsilon))^{-1/2}, & \beta = d - 1, \\ \varepsilon^{-1/2}, & \beta < d - 1. \end{cases}$$

Lemma 1. *Let $\tau(\varepsilon)$ and H be as in Theorem 4. For $M \geq 0$ and $t \in \mathbb{R}^d$, define*

$$(33) \quad \tilde{\Psi}(M, t) = \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \int_{\mathbb{R}^d} \int_0^M |\delta_{\varepsilon t}(B(x, u)) - \delta_0(B(x, u))| u^{-\beta-1} du dx.$$

Then $\tilde{\Psi}(M, t) = c |t|^{2H}$, where $0 \leq c < \infty$ is a constant depending on M and β but independent of t .

Proof. By a change of the order of integration in (33) we obtain

$$\begin{aligned} \tilde{\Psi}(M, t) &= \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \int_0^M u^{-\beta-1} |B(\varepsilon t, u) \triangle B(0, u)| du \\ &= \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \int_0^M u^{d-\beta-1} |B(\varepsilon t/u, 1) \triangle B(0, 1)| du. \end{aligned}$$

Hence,

$$\tilde{\Psi}(M, t) = \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \varepsilon^{d-\beta} |t|^{d-\beta} \int_0^{\frac{M}{\varepsilon|t|}} u^{d-\beta-1} |B(e/u, 1) \triangle B(0, 1)| du$$

for some $e \in \mathbb{S}^{d-1}$. Using the function $V(u)$ in (11) for the volume of the intersection of two balls of radius 1 and center distance u ,

$$|B(e/u, 1) \triangle B(0, 1)| = \begin{cases} 2V(0), & u \leq 1/2, \\ 2(V(0) - V(1/u)), & u > 1/2. \end{cases}$$

By L'Hospital's rule

$$\lim_{u \rightarrow \infty} u |B(e/u, 1) \triangle B(0, 1)| = \lim_{u \rightarrow \infty} 4\nu_{d-1} u \int_0^{1/(2u)} (1 - s^2)^{\frac{d-1}{2}} ds = 2\nu_{d-1}$$

and so

$$\tilde{\Psi}(M, t) = \text{const} \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \varepsilon^{d-\beta} |t|^{d-\beta} \int_0^{M/(\varepsilon|t|)} u^{d-\beta-1} \min\{1, u^{-1}\} du.$$

By evaluating this integral expression separately for the three different intervals of β and the corresponding scaling functions $\tau(\varepsilon)$, we obtain

$$\tilde{\Psi}(M, t) = \text{const} |t|^{2H}$$

for the choice the of Hurst index stated in Theorem 4. □

Proof of Theorem 4. We note that the limit of a sequence of Gaussian processes is Gaussian and that Gaussian processes are determined by their two-dimensional distributions. Fix an element $z \in D$. We define formally

$$(34) \quad \begin{aligned} T'(t) &= \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon) (W_\beta(z + \varepsilon t) - W_\beta(z)) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d \times \mathbb{R}_+} \tau(\varepsilon) (\delta_{z+\varepsilon t} - \delta_z)(B(x, u)) M_\beta(dx, du). \end{aligned}$$

Then the covariance of T' is given by

$$\mathbb{E}T'(s)T'(t) = \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \int_{\mathbb{R}^d \times \mathbb{R}_+} (\delta_{z+\varepsilon s} - \delta_z)(\delta_{z+\varepsilon t} - \delta_z)(B(x, u)) \nu^D(dx, du).$$

We have

$$(\delta_{z+\varepsilon s} - \delta_z)(\delta_{z+\varepsilon t} - \delta_z) = \frac{1}{2}((\delta_{z+\varepsilon s} - \delta_z)^2 + (\delta_{z+\varepsilon t} - \delta_z)^2 - (\delta_{z+\varepsilon s} - \delta_{z+\varepsilon t})^2).$$

Moreover,

$$(\delta_{z+\varepsilon s} - \delta_{z+\varepsilon t})(B(x, u)) = (\delta_{z+\varepsilon s-\varepsilon t} - \delta_z)(B(x - \varepsilon t, u)),$$

and thus

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}_+} (\delta_{z+\varepsilon s} - \delta_{z+\varepsilon t})^2 (B(x, u)) u^{-\beta-1} \mathbb{I}(B(x, u) \subset D) dx du \\ &= \int_{\mathbb{R}^d \times \mathbb{R}_+} (\delta_{z+\varepsilon(s-t)} - \delta_z)^2 (B(x, u)) u^{-\beta-1} \mathbb{I}(B(x, u) \subset D - \varepsilon t) dx du. \end{aligned}$$

We obtain

$$(35) \quad \mathbb{E}T'(s)T'(t) = \frac{1}{2} (\Psi(s, D) + \Psi(t, D) - \Psi(s - t, D - \varepsilon t))$$

with

$$\begin{aligned} \Psi(t, D) &= \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \int_{\mathbb{R}^d \times \mathbb{R}_+} (\delta_{z+\varepsilon t} - \delta_z)^2 (B(x, u)) u^{-\beta-1} \mathbb{I}(B(x, u) \subset D) dx du \\ &= \lim_{\varepsilon \rightarrow 0} \tau(\varepsilon)^2 \int_{\mathbb{R}^d \times \mathbb{R}_+} (\delta_{\varepsilon t} - \delta_0)^2 (B(x, u)) u^{-\beta-1} \mathbb{I}(B(x, u) \subset D - z) dx du. \end{aligned}$$

We evaluate $\Psi(t, D)$. The third term in (35) then requires only small modifications because we have to work with $D - \varepsilon t$ instead of D . Recalling the definition of $\tilde{\Psi}(M, t)$ in (33), $\tilde{\Psi}(\cdot, t)$ is a continuous, monotone increasing function with $\tilde{\Psi}(0, t) = 0$. Since D is bounded there is an $N > 0$ which only depends on D such that $\mathbb{I}(B(x, u) \subset D) = 0$ for all $u > N$ and $x \in \mathbb{R}^d$. Hence, $\Psi(t, D) \leq \tilde{\Psi}(N, t)$. A careful reading of the proof of Lemma 1 shows that, for any $v \in \mathbb{R}$, we may replace M by $M + \varepsilon v$ in the right hand side of (33) without changing the constant in $\tilde{\Psi}(M, t) = \text{const } |t|^{2H}$. Therefore we can find an M , $0 \leq M \leq N$, depending on D , β , and z , but independent of t , such that $\tilde{\Psi}(M, t) = \Psi(t, D)$. Hence by Lemma 1, $\Psi(t, D) = c |t|^{2H}$, with a constant c independent of t . It follows immediately that $\Psi(s, D) = c |s|^{2H}$ and $\Psi(s - t, D - \varepsilon t) = c |s - t|^{2H}$, and hence

$$\mathbb{E}T'(s)T'(t) = c (|s|^{2H} + |t|^{2H} - |s - t|^{2H}). \quad \square$$

The hard boundary thinning bridge on $[0, T]$. We consider the Gaussian membrane W_β for the special case $d = 1$ and $D = (0, T)$, some $T > 0$. By (29),

$$\begin{aligned}\mathbb{E}W_\beta(s)W_\beta(t) &= \int_0^T \int_0^\infty \mathbb{I}(0 < x - u < s, t < x + u < T) u^{-\beta-1} du dx \\ &= f_\beta(s \vee t) + f_\beta(T - s \wedge t) - f_\beta(|s - t|) - f_\beta(T),\end{aligned}$$

where

$$f_\beta(x) = \begin{cases} \frac{2^\beta}{\beta(1-\beta)} x^{1-\beta}, & \beta < d, \beta \neq 0, \\ -x \ln x, & \beta = 0. \end{cases}$$

We point out that the case $\beta = -1$ is the classical Brownian bridge on $[0, T]$ with covariance function

$$\mathbb{E}W_{-1}(s)W_{-1}(t) = \frac{1}{2} s \wedge t (T - s \vee t).$$

For any other value of the parameter β , however, the hard boundary Gaussian bridge is different from the fractional Brownian bridge on $[0, T]$ obtained as fractional Brownian motion pinned to zero at time T . To conclude we provide an additional result on the relation between the hard boundary bridge and fractional Brownian motion.

Proposition 2. *Let $B = (B_t)_{t \in [0, T]}$ be standard linear Brownian motion independent from W_β and define the Gaussian martingale $Y_\beta = (Y_\beta(t))_{t \in [0, T]}$ by*

$$(36) \quad Y_\beta(t) = \sqrt{\frac{2^\beta}{\beta}} \int_0^t \sqrt{x^{-\beta} + (T - x)^{-\beta}} dB_x.$$

Then, for $0 < \beta < 1$, $W_\beta + Y_\beta$ is (up to constant) a fractional Brownian motion with Hurst index $H = (1 - \beta)/2$.

Proof. Put $Z_\beta = W_\beta + Y_\beta$. As the sum of two independent Gaussian processes, Z_β is a Gaussian process as well. Thus, it is enough to show that the covariance $\mathbb{E}Z_\beta(s)Z_\beta(t)$ is given by a multiple of $C_\beta(s, t)$, with

$$C_\beta(s, t) = s^{1-\beta} + t^{1-\beta} - |s - t|^{1-\beta}.$$

It is easily seen that

$$\mathbb{E}W_\beta(s)W_\beta(t) = \frac{2^\beta}{\beta(1-\beta)} (C_\beta(s, t) - C_\beta(s \wedge t, T)).$$

Since $\mathbb{E}Z_\beta(s)Z_\beta(t) = \mathbb{E}W_\beta(s)W_\beta(t) + \mathbb{E}Y_\beta(s)Y_\beta(t)$ for all $s, t \in [0, T]$, it is enough to show that $\mathbb{E}Y_\beta(s)Y_\beta(t) = 2^\beta C_\beta(s \wedge t, T)/(\beta(1-\beta))$. In fact,

$$\begin{aligned}\mathbb{E}Y_\beta(s)Y_\beta(t) &= \frac{2^\beta}{\beta} \int_0^{s \wedge t} x^{-\beta} + (T - x)^{-\beta} dx \\ &= \frac{2^\beta}{\beta(1-\beta)} \left((s \wedge t)^{1-\beta} - (T - s \wedge t)^{1-\beta} + T^{1-\beta} \right) \\ &= \frac{2^\beta}{\beta(1-\beta)} C_\beta(s \wedge t, T).\end{aligned}$$

□

We may extend the definition of Y_β in (36) to $-1 < \beta < 0$. Then Y_β becomes a purely imaginary Gaussian process. Setting $Z_\beta = W_\beta + Y_\beta$ as before yields a complex (centered) Gaussian process with

$$\mathbb{E}Z_\beta(s)Z_\beta(t) = \mathbb{E}Z_\beta(s)\overline{Z_\beta(t)} = \frac{2^\beta}{\beta(1-\beta)}C_\beta(s, t).$$

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